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"Source Structure Simulation"

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Source Structure Simulation

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1. Introduction

The change in interferometer phase as a function of frequency will affect the possibility of connecting phase across a large frequency range, such as 2 GHz to 15 GHz, as proposed for a means of obtaining good delay precision at low signal to noise ratio (SNR).

2. Visibility function

The visibility function is the Fourier Transform (FT) of the brightness distribution of a radio source. For a model described in terms of Gaussian components the visibility function is simple to calculate, since the FT of a Gaussian is also a Gaussian.

Describe each Gaussian component, *i*, by six parameters:

$S_i =$ flux density (Jy)

$\theta_{\alpha i} =$ half-power width in RA (or major axis)

$\theta_{\delta i} =$ half-power width in dec (or minor axis)

$\alpha_i =$ offset in RA

$\delta_i =$ offset in dec

$\phi_i =$ position angle of major axis (ccw from δ-axis)

The area under a one-dimensional Gaussian is

$$\frac{a}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-a^2x^2)\,dx = 1$$

and the Gaussian in terms of the full width at half amplitude, θ_H, is

$$f(x) = C\exp(-4\ln 2(x^2/\theta_H^2))$$

Then the coefficient of the normalized (integral = 1) function is

$$C = \frac{1}{\theta_H}\sqrt{\frac{4\ln 2}{\pi}}$$

So the function describing the two-dimensional Gaussian component having total flux density S_i, allowing for different sizes along the two axes, is

$$I_i(\alpha,\delta) = S_i\frac{4\ln 2}{\pi\theta_{\alpha i}\theta_{\delta i}}\exp(-4\ln 2((\alpha-\alpha_i)^2/\theta_{\alpha i}^2)+(\delta-\delta_i)^2/\theta_{\delta i}^2)) \qquad (2.1)$$

This assumes that the major axis of the component lies along one of the axes of the coordinate system. The description for arbitrary position angle will be addressed later.

The Fourier transform of

$$f(x) = \exp(-a^2 x^2)$$

is

$$F(u) = \frac{\sqrt{\pi}}{a}\exp(-\pi^2 u^2 / a^2)$$

So the complex visibility function, $\Gamma(u,v)$, is

$$\Gamma_i(u,v) = S_i \exp\left\{-\frac{\pi^2}{4\ln 2}(u^2\theta_{\alpha i}^2 + v^2\theta_{\delta i}^2) + j2\pi(u\alpha_i + v\delta_i)\right\} \tag{2.2}$$

with cos and sin terms (for real and imaginary)

$$\begin{aligned} \Gamma_{ic}(u,v) &= S_i \exp\left\{-\frac{\pi^2}{4\ln 2}(u^2\theta_{\alpha i}^2 + v^2\theta_{\delta i}^2)\right\}\cos\{2\pi(u\alpha_i + v\delta_i)\} \\ \Gamma_{is}(u,v) &= S_i \exp\left\{-\frac{\pi^2}{4\ln 2}(u^2\theta_{\alpha i}^2 + v^2\theta_{\delta i}^2)\right\}\sin\{2\pi(u\alpha_i + v\delta_i)\} \end{aligned} \tag{2.3}$$

I have used the *j* notation for the imaginary part since *i* is used as the component index.

The cos and sin terms will be summed separately at each *(u,v)* for all of the components. The visibility amplitude and phase are then given by

$$\begin{aligned} \Gamma(u,v) &= \left[\left(\sum_i \Gamma_{ic}(u,v)\right)^2 + \left(\sum_i \Gamma_{is}(u,v)\right)^2\right]^{1/2} \\ \Phi(u,v) &= \tan^{-1}\left(\sum_i \Gamma_{is}(u,v) / \sum_i \Gamma_{ic}(u,v)\right) \end{aligned} \tag{2.4}$$

3. Coordinate system

For the sky coordinate system define declination as positive up on the sky (and on the page for visualization) and right ascension as positive to the left (as though lying on your back looking up). Then angles are measured counter-clockwise from positive declination (right hand rule).

4. Visibility calculation

For an elliptical Gaussian source component that is not aligned with the RA or dec axis, calculate the visibility by projecting the baseline (*u,v*) on the major and minor axes of the component transform. Thus in the component (*u',v'*) frame, with *u'* along the major axis

$$\begin{aligned} u_i' &= u\sin(\phi_i) + v\cos(\phi_i) \\ v_i' &= u\cos(\phi_i) - v\sin(\phi_i) \end{aligned} \tag{4.1}$$

The argument of the exponential then becomes

$$\arg 1_i(u,v) = -\frac{\pi^2}{4\ln 2}((u'\theta_{\alpha i})^2 + (v'\theta_{\delta i})^2) \quad (4.2)$$

5. Intensity calculation

Similar to the calculation of the visibility, calculate the intensity for an elliptical Gaussian source component that is not aligned with the RA or dec axis by projecting the (α,δ) axes on the major and minor axes of the component. Thus in the component (α',δ') frame, with α' along the major axis

$$\begin{aligned} \alpha_i' &= (\alpha - \alpha_0)\sin(\phi_i) + (\delta - \delta_0)\cos(\phi_i) \\ \delta_i' &= (\alpha - \alpha_0)\cos(\phi_i) - (\delta - \delta_0)\sin(\phi_i) \end{aligned} \quad (5.1)$$

So the function describing the two-dimensional Gaussian component having total flux density S_i, allowing for different sizes along the two axes, is

$$I_i(\alpha,\delta) = S_i \frac{4\ln 2}{\pi\theta_{\alpha i}\theta_{\delta i}} \exp(-4\ln 2((\alpha'/\theta_{\alpha i})^2 + (\delta'/\theta_{\delta i})^2)) \quad (5.2)$$

In order to calculate the phase across the full frequency range, the structure should be defined as a function of frequency. To achieve this I have simplified the spectrum of each component to four parameters: maximum flux density, frequency of that value, and spectral indices below and above that frequency. To complete the description of each component, five more parameters are required: distance and position angle from the origin, major axis, axial ratio, and position angle of the major axis. These five, plus the flux density calculated from the spectrum, correspond to the six parameters of section 2. As initial values for the spectral indices I have used 2.5 and -0.5 below and above the peak frequency.

6. *matlab* scripts

The source structure is depicted as a contour plot using *plot_source.m*. The component size is not convolved with a restoring beam at this time (06/08/08). Calculation of the visibilities on a range of baselines at a fixed frequency is also incorporated in this script.

Separate calculation of the component spectra and source visibilities is implemented in two *matlab* scripts:

vis_by_freq_1.m: calculate and plot complex visibilities for fixed baseline across frequency band 2 GHz – 16 GHz. The visibility amplitude and phase can be written to file from this script.

vis_by_bsln_1.m: calculate and plot complex visibilities for fixed frequency for baselines from 0 to 10000 km.

7. Example

As an illustration of these scripts I have constructed a model covering the range 2 GHz to 16 GHz for the source 0248+430 based on the 2.3 and 8.5 GHz maps and Gaussian components from Fey and Charlot (2000). Since there are values of component flux density at only two frequencies, the spectra are constrained only to give approximately those two values. For example, the peak frequency of some components could be moved higher and compensated for by allowing a more negative spectral index above that frequency. Additional maps at 5 GHz or 15 GHz would improve the model.

The model is given in the file *0248p430*:

```
%  S0    f0    alfa_a alfa_b  r_comp pa_comp  size ax_ratio  pa_ax
% (Jy) (GHz)                   (mas)   (deg)  (mas)            (deg)
  1.0  2.0  2.5     -0.5    0.0    0.0     1.5    0.2 -40.0
  0.25 2.0  2.5     -0.5    1.5  135.0     0.4    0.1 135.0
  0.1  1.0  2.5     -0.7    5.0  135.0     0.5    1.0 135.0
  0.3  1.0  2.5     -0.7   12.0  143.0     1.35   1.0   0.0
  0.03 1.0  2.5     -0.7   26.0  166.0     9.0    1.0   0.0
```

The spectral decomposition is illustrated in Figure 1a and the contour plot in 1b. The contour levels are at 2^{-n}, with n = 0 – -20 of the maximum. Such a large value is needed in order to see the contours of the fifth component. The visibility amplitude and phase are shown in Figure 2 as a function of frequency for a baseline of 6400 km. In Figure 3 the phases at 2.3 GHz and 12 GHz are shown for baselines from 0 to 10000 km.

The increase in phase with increasing frequency above about 6 GHz (Figure 2b) is possibly due to the change in apparent mean position from close to component 1 towards component 2 as component 1 is resolved. This intuitive conjecture could be checked by calculating mean apparent position as a function of frequency using the visibility amplitudes of the components (which are calculated in the scripts).

8. References

Fey, A., and P. Charlot, VLBA Observation of radio reference frame sources. III. Astrometric suitability of an additional 225 sources, Ap. J. Supp. Series, **128**:17-83, 2000 May.

Thompson, Moran, and Swenson, Interferometry and Synthesis in Radio Astronomy, Second Edition, John Wiley and Sons, Inc., 2001.

9. Figures

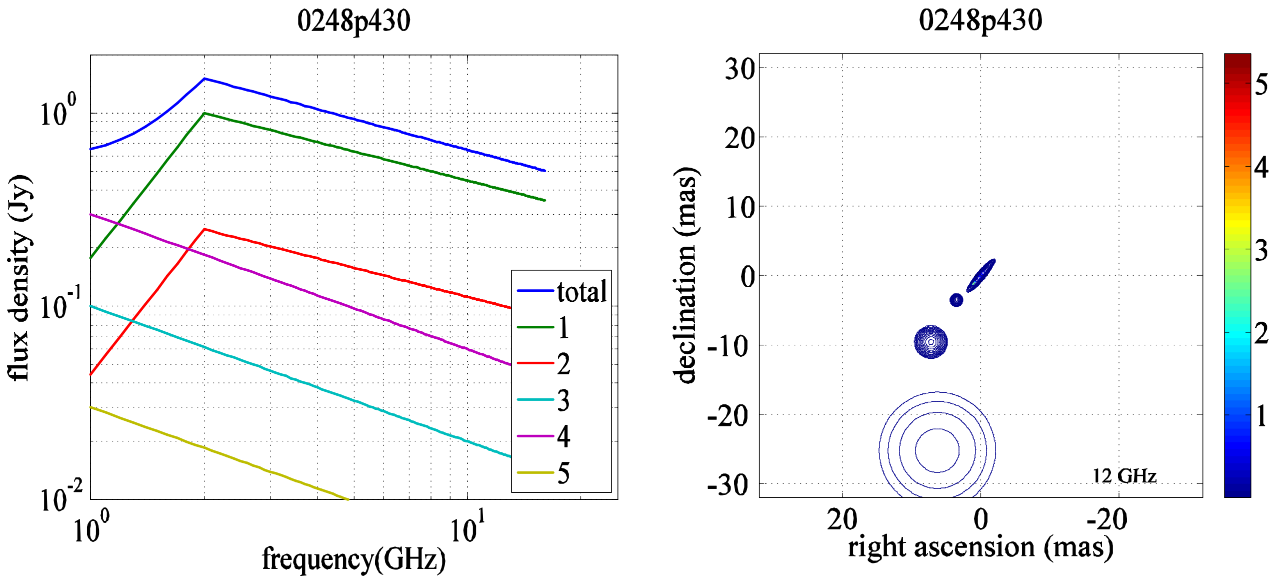

Figure 1: a) Contour plot of the sum of the components of 0248+483 listed in section 6. Contour levels are in decreasing powers of 2 from 0 to -20. b) Spectra of the five components and total flux density. At 3 GHz the order from the top down is total, 1, 2, 4, 3, 5.

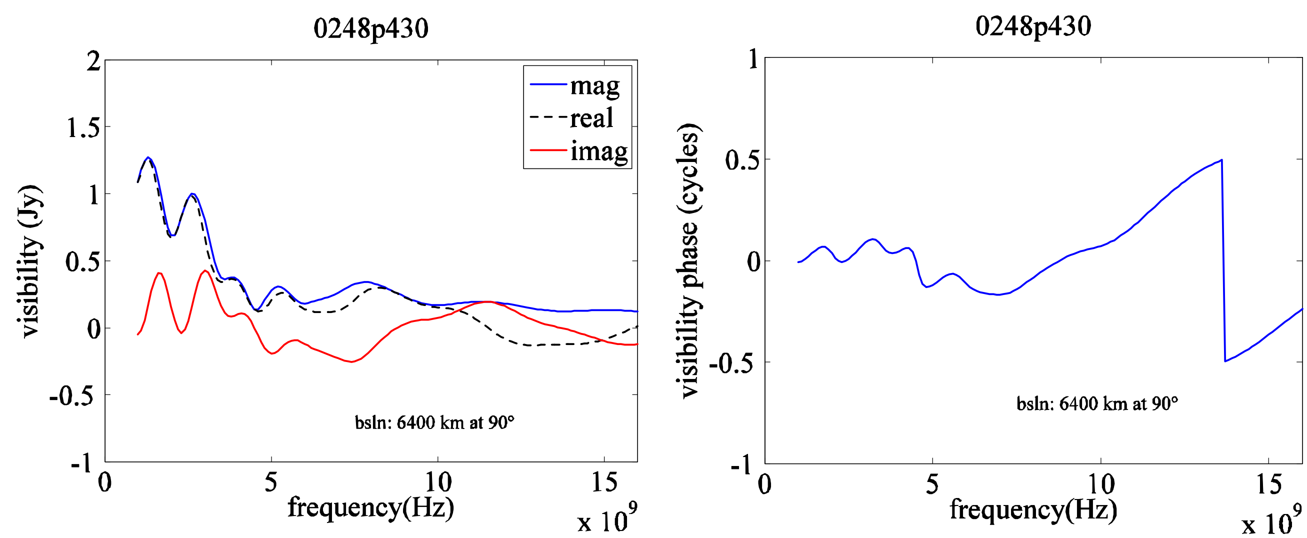

Figure 2. a) Visibility (correlated flux density) as a function of frequency for an equatorial baseline of 6400 km. The top solid line (blue) is the magnitude of the visibility. b) Phase in cycles.

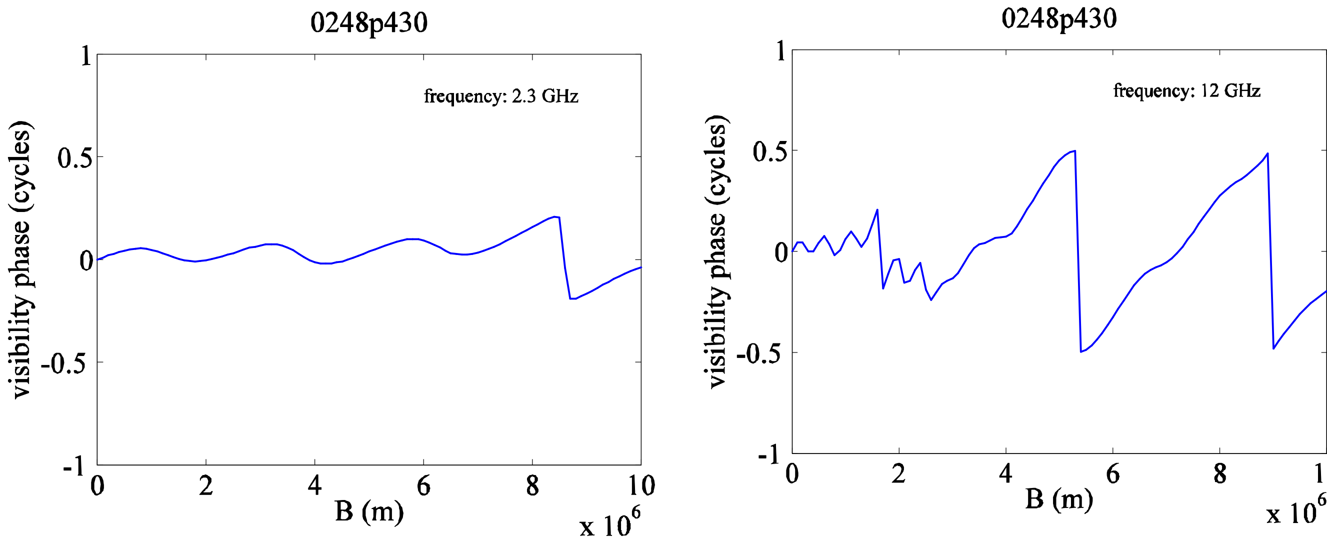

Figure 3. Visibility phase by baseline length for a) 2.3 GHz and b) 12 GHz.