To: Ad Hoc Working Group on HF-EOP

From: Richard Ray (GSFC)

Re: Indexing and argument conventions for tides

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**Summary**

I would like to urge this working group—and hopefully eventually the wider tidal EOP community—to adopt tidal indexing and tidal argument conventions that are more consistent with those commonly used by the Earth tide and ocean tide communities. More specifically I urge adoption of a convention that follows more closely Doodson's elegant 1921 scheme, slightly extended.

An immediate advantage of doing this would be that advances in ocean-tide modeling can be more readily converted into new models for the geodetic EOP community, without the on-going confusions and errors that occur now. But aside from the issue of consistency among different communities, there are other reasons to use standard ocean-tide conventions, since they more readily help us understand underlying physical differences among models. Current EOP models in the form now tabulated by the IERS tempts many users to think of these models as black boxes of random numbers. One could thus even argue there are strong *aesthetic* reasons for following Doodson.


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What I will here call the Woolard convention dates back to the 1950s and to his development of the tidal potential for use by the nutation community[1]. Woolard (1953) expressed his tidal arguments as linear combinations of the fundamental variables that E. W. Brown (and others earlier) employed in his lunar theory. The tidal arguments adopted by Woolard (1959) are closely related. In my humble opinion, Doodson's earlier convention is much preferable. As is well known, Doodson, who also used Brown's lunar theory, re-expressed his final series in terms of six fundamental variables, all related to those used by Woolard. One great advantage of Doodson's system stems from the very different temporal rates of the six variables (see Table 1),

[1] I'll be indebted to anyone wishing to correct my (limited) historical understanding of how we ended up with the present conventions.

Table 1: Fundamental variables—Doodson versus Delaunay

| | | Rate (cpd) | Period | |
|---|---|---|---|---|
| $\tau$ | mean lunar time | $9.661 \times 10^{-1}$ | 1.03505 d | (lunar day) |
| $s$ | mean longitude of moon | $3.660 \times 10^{-2}$ | 27.32158 d | (tropical month) |
| $h$ | mean longitude of sun | $2.738 \times 10^{-3}$ | 365.2422 d | (tropical year) |
| $p$ | mean longitude of lunar perigee | $3.095 \times 10^{-4}$ | 8.847 y | (lunar orbit precession) |
| $N'$ | negative longitude of lunar node | $1.471 \times 10^{-4}$ | 18.61 y | (regression of lunar node) |
| $p_s$ | mean longitude of solar perigee | $1.307 \times 10^{-7}$ | 21,000 y | |
| $\gamma$ | Greenwich mean sidereal time | $1.003 \times 10^{0}$ | 0.99727 d | (sidereal day) |
| $l$ | mean anomaly of moon | $3.629 \times 10^{-2}$ | 27.5545 d | (anomalistic month) |
| $l'$ | mean anomaly of sun | $2.738 \times 10^{-3}$ | 365.2596 d | (anomalistic year) |
| $F$ | $s - \Omega$ | $3.675 \times 10^{-2}$ | 27.2122 d | (draconic month) |
| $D$ | mean elongation of moon | $3.386 \times 10^{-2}$ | 29.5306 d | (synodic month) |
| $\Omega$ | mean longitude of lunar node | $-1.471 \times 10^{-4}$ | 18.61 | (regression of lunar node) |

so that when tidal lines are tabulated by frequency, the integer expansion coefficients fall into a simple, orderly pattern. Or as Doodson himself wrote:

> "it is a curious fact that if we classify in terms of $\tau$ [lunar mean time], with a sub-classification with regard to $s$ [the moon's mean longitude], and a further sub-classification with regard to $h$ [the sun's mean longitude], the constituents are completely separated into groups with no over-lapping of speeds [frequencies]. It is still more curious that to the order required the same process can be continued for all the variables. Owing to this, a rather elegant and very useful form of presentation of the results is possible" (Doodson, 1921).

Let us write the integer Doodson argument indices as the 6-integer set $(k_1, k_2, \ldots\ k_6)$ (I won't use here the often-added '5's, which were merely a bookkeeping convenience in the days of hand tabulations). To this standard Doodson argument I find that adding a seventh index, $k_7$, being a simple integer multiple of 90°, is useful and makes the whole argument clearer to everyone. This multiple of 90° then permits the cosine function to be used consistently with all arguments and ensures all amplitudes are positive.[2] Sometimes this additional $k_7$ is left unmentioned, which tends

[2]The original algebraic expansion of the tidal potential by Doodson led to both positive and

to confuse non-experts. Many modern texts (e.g., Pugh & Woodworth, 2014, Table 4.1[3]) do include this extra multiple of 90°, at least implicitly, if not explicitly as the additional $k_7$.

To see how this plays out, consider Table 2 where the UT1 coefficients of a few of the major diurnal constituents are listed (by frequency) in the two different conventions. As Doodson's quotation above stresses, his integer indices fall in a clear orderly pattern, automatically sorting the constituents by frequency. The Woolard indices, aside from the first one that denotes a diurnal wave, appear in almost random order.

Or consider the two spectral lines that make up the (gravitational) $S_1$ constituent (I didn't include these in Table 2). In Doodson's scheme we have

| $\tau$ | $s$ | $h$ | $p$ | $N'$ | $p_s$ |
|---|---|---|---|---|---|
| 1 | 1 | -1 | 0 | 0 | -1 |
| 1 | 1 | -1 | 0 | 0 | 1 |

Anyone looking at these indices would know immediately that the two spectral lines are of almost identical frequencies, differing only by 2 cycles in 21,000 years. The Woolard scheme for the two $S_1$ lines is

| $\gamma$ | $l$ | $l'$ | $F$ | $D$ | $\Omega$ |
|---|---|---|---|---|---|
| 1 | 0 | 1 | -2 | 2 | 2 |
| 1 | 0 | -1 | 0 | 0 | 0 |

I defy anyone to say they can look at these indices and know immediately that the two lines have nearly the same frequency!

The numerical coefficients themselves also are more easily interpretable in the Doodson scheme. In Table 2, all sine and cosine coefficients have the same sign across the band, except at the edge of the band where the sine component of $OO_1$ turns negative. All constituents have roughly similar phase lags on the potential, of order 30° or so, until again at the high-frequency part of the band above $\theta_1$ where the phases become very small and then turn negative at $OO_1$. Dividing these COS,SIN coefficients by the magnitude of each constituent's driving potential yields an "admittance" $(Z_r, Z_i)$ which I've tacked on to the table in the last two columns. In Doodson's scheme, the admittances are seen to trace out a (mostly) smooth function of frequency, a fact that Richard Eanes tried to exploit when he urged adoption of the

negative amplitudes, and sine functions for diurnal waves; Cartwright & Tayler followed this, although their expansion was done numerically; the expansion of Yoder et al. (1981) also has negative amplitudes. The 90° phase augmentations thus accounts for all this, allowing us to consistently use cosine functions throughout.

[3]But watch for a typographical error in Pugh-Woodworth's argument of $K_1$.

Table 2: Coefficients of selected diurnal UT1 constituents ($\mu$s)*

| Woolard convention (IERS, 2010) | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Tide | $\gamma$ | $l$ | $l'$ | $F$ | $D$ | $\Omega$ | Cos | Sin | $Z_r$ | $Z_i$ |
| $\sigma_1$ | 1 | 0 | 0 | -2 | -2 | -2 | $-0.39$ | 1.19 | $-0.46$ | 1.50 |
| $Q_1$ | 1 | -1 | 0 | -2 | 0 | -2 | $-2.50$ | 5.12 | $-0.50$ | 1.02 |
| $\rho_1$ | 1 | 1 | 0 | -2 | -2 | -2 | $-0.47$ | 0.91 | $-0.47$ | 0.94 |
| $O_1$ | 1 | 0 | 0 | -2 | 0 | -2 | $-12.07$ | 16.02 | $-0.46$ | 0.61 |
| $M_1$ | 1 | -1 | 0 | 0 | 0 | 0 | 0.75 | $-0.86$ | 0.37 | $-0.43$ |
| $P_1$ | 1 | 0 | 0 | -2 | 2 | -2 | $-3.10$ | 5.51 | $-0.25$ | 0.44 |
| $K_1$ | 1 | 0 | 0 | 0 | 0 | 0 | 8.55 | $-17.62$ | 0.23 | $-0.48$ |
| $\theta_1$ | 1 | -1 | 0 | 0 | 2 | 0 | 0.04 | $-0.29$ | 0.08 | $-0.71$ |
| $J_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0.19 | $-1.61$ | 0.08 | $-0.75$ |
| $OO_1$ | 1 | 0 | 0 | 2 | 0 | 2 | $-0.04$ | $-1.44$ | $-0.02$ | $-1.26$ |

| Doodson convention (Chao, 1996) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Tide | $\tau$ | $s$ | $h$ | $p$ | $N'$ | $p_s$ | 90° | Cos | Sin | $Z_r$ | $Z_i$ |
| $\sigma_1$ | 1 | -3 | 2 | 0 | 0 | 0 | -1 | 1.21 | 0.37 | 1.51 | 0.47 |
| $Q_1$ | 1 | -2 | 0 | 1 | 0 | 0 | -1 | 5.03 | 2.45 | 1.00 | 0.49 |
| $\rho_1$ | 1 | -2 | 2 | -1 | 0 | 0 | -1 | 0.89 | 0.47 | 0.94 | 0.49 |
| $O_1$ | 1 | -1 | 0 | 0 | 0 | 0 | -1 | 16.05 | 12.10 | 0.61 | 0.46 |
| $M_1$ | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0.90 | 0.78 | 0.44 | 0.38 |
| $P_1$ | 1 | 1 | -2 | 0 | 0 | 0 | -1 | 5.16 | 2.86 | 0.42 | 0.23 |
| $K_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 17.71 | 8.64 | 0.48 | 0.23 |
| $\theta_1$ | 1 | 2 | -2 | 1 | 0 | 0 | 1 | 0.27 | 0.03 | 0.69 | 0.07 |
| $J_1$ | 1 | 2 | 0 | -1 | 0 | 0 | 1 | 1.52 | 0.08 | 0.74 | 0.04 |
| $OO_1$ | 1 | 3 | 0 | 0 | 0 | 0 | 1 | 1.38 | $-0.24$ | 1.22 | $-0.22$ |

* The same tide model is displayed in the two indexing conventions. The numerical coefficients differ slightly because the bottom table takes the coefficients directly from the paper of Chao et al. (1996), whereas the IERS coefficients went through various transformations, in and out of orthotides.

response (orthotide) formalism for EOP tables; this smoothness across the band in the tabulated $(Z_r, Z_i)$ parameters is thanks to the logic behind the Doodson phases.

Finally, it is worth noting that some standard nomenclature for tides makes sense only within the confines of Doodson's scheme. Consider the technical definitions of the terms species, group, and constituent: All terms with the same value of $k_1$ constitute a tidal "species," with different species separated in frequency by about one cycle per lunar day. All terms with the same $(k_1, k_2)$ constitute a tidal "group," with different groups separated by about one cycle per month. And all terms with the same $(k_1, k_2, k_3)$ constitute a tidal "constituent," with different constituents separated by one cycle per year. (Thus, despite the loose terminology often seen in the literature, a tidal constituent consists in general of a cluster of spectral lines, not a single isolated line.) As Table 2 shows, the identification of tidal species, groups, and constituents becomes automatic and corresponds nicely to the layout of the Doodson-based table.

To those who argue that the Woolard convention is now firmly entrenched within the EOP community and things should therefore not be upended, I would argue that this may be true within the nutation community, but not the geodetic EOP community. In fact, the paper on which the current IERS-2010 model of EOP is based (Chao et al., 1996) uses the Doodson convention. Moreover, those worried about overturning traditions should consider Woolard's original work on the tidal expansion, which was less comprehensive and (as argued above) less elegant than Doodson's work done some thirty years earlier. Woolard (1959) does acknowledge Doodson's earlier work, when he noted that his own expansion could be checked by comparing it against Doodson's. But he did reinvent the wheel.

## References


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