An Alternative Model of the Gravitational Delay

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Abstract A massive body causes a time delay in the gravitational field of a body. The conventional gravitational delay model was expanded into a Taylor series on $O(\frac{b}{r})$. The results of the expression show a dependence on the positions of the two antennas, the observed source (quasar), and the massive body (Sun or Jupiter). In this paper we compare the Taylor series expansion approach with the conventional gravitational delay. The total difference of these models stays below the accuracy limit.

Keywords VLBI, gravitational delay model, general relativity

1 Introduction

The conventional gravitational delay model for the reduction of geodetic VLBI [1] observations was developed in the 1980s and its use was recommended by the IAU General Assembly in 1991. We propose an alternative model by using the Taylor series expansion of the gravitational delay on the small parameter. A numerical comparison of these two models shows their consistency at the 1 picosecond (ps) level for the Sun's gravitational field (at an angular distance of about 4°). Although the new formula is more complex, because it comprises several terms, there are some advantages: 1) direct analytical link to the effect of the light deflection, 2) exclusion of the coordinate terms (total potential of the solar system bodies) in the gravitational and geometric delays, and 3) applicability to a wide selection of related effects.

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2 The Conventional Formula

Besides the three classical tests, the fourth test of General Relativity—the delay of a signal propagating in the gravitational field—has been proposed by Shapiro [2] and is known as the Shapiro delay. The difference between the two Shapiro delays as measured with two radio telescopes gives a gravitational delay which must be considered at the standard reduction of the high-precision geodetic VLBI data. The IERS Conventions 2010 [3] contain the conventional formula for the gravitational delay, which is valid for most cases unless a distant quasar and a deflecting body are too close. This formula is presented as follows

$$\tau_{grav} = \frac{(\gamma+1)GM}{c^3} \ln \frac{|\mathbf{r_1}| + \mathbf{s}\cdot\mathbf{r_1}}{|\mathbf{r_2}| + \mathbf{s}\cdot\mathbf{r_2}}, \qquad (1)$$

where γ is the PPN-parameter of General Relativity [4], G is the gravitational constant, M is the mass of gravitational body, c is the speed of light, $\mathbf{s}$ is the barycentric unit vector towards the radio source, and $\mathbf{r_i}$ is the vector between the center of mass of the gravitating body and the i-th telescope.

3 Two Examples of Geodetic VLBI Sessions

One close approach of Jupiter to the radio source 1922–224 on 18/19 November 2008 was observed during the OHIG60 session. Four stations (Hobart26, Kokee, Tsukub32, and Parkes) were tracking the radio source over twelve hours. The minimum approach distance was $1.4'$. A special "hand-made" schedule was prepared by Dirk Behrend (GSFC).

Several radio sources were observed in close angular proximity to the Sun during the R&D session RD1208 on 2/3 October 2012. The radio source 1243–072 was tracked at the range of angular distance from of $3.7°$ to $4.3°$. Five stations (Kokee, Tsukub32, HartRAO, Wettzell, and Onsala60) participated in this session.

4 General Relativity Delay Model

An expression that links the gravitational delay and the formula for the light deflection angle [5] is yet to be developed. To obtain it we have expanded the gravitational delay using a Taylor series expansion on $O(\frac{b}{r})$. We keep the terms of order $(\frac{b}{r})^2$ that correspond to an accuracy of about 1 ps.

We expand Formula (1) as a series of $\frac{b}{r_2}$ where r_2 is the barycentric vector of the second station:

$$\begin{aligned}\tau_{grav} = &-\frac{(\gamma+1)GM}{c^3}\frac{b}{r_2}\cos\varphi && \{\text{coordinate term}\}\\ &+\frac{(\gamma+1)GM}{c^3}\frac{b}{r_2}\frac{\sin\varphi\sin\theta\cos A}{1-\cos\theta} && \{\text{term } t_1\}\\ &-\frac{(\gamma+1)GM}{2c^3}\left(\frac{b\sin\varphi\sin\theta\cos A}{r_2(1-\cos\theta)}\right)^2 && \{\text{term } t_2\}\\ &+\frac{(\gamma+1)GM}{2c^3}\frac{b^2}{r_2^2}\frac{1-\cos^2\varphi\cos^2\theta}{1-\cos\theta} && \{\text{term } t_3\},\end{aligned} \tag{2}$$

where the vectors $\mathbf{b}$ and $\mathbf{r}$ and the angles φ, ψ, θ, and A are shown in Figure 1. In order to provide more accurate modeling of the gravitational delay, in the calculations one has to use the vector $\mathbf{r_2}$ for the second station instead of the barycentric vector $\mathbf{r}$ of Figure 1.

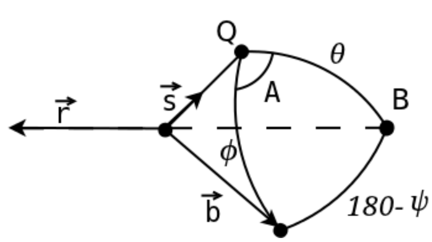

Fig. 1 Angle θ – the impact parameter, angle φ between vectors $\mathbf{b}$ and $\mathbf{s}$, and angle ψ between vectors $\mathbf{b}$ and $\mathbf{r}$.

Figure 1 shows the positions of the quasar Q, the deflecting body B (e.g., Jupiter or Sun), the baseline vector $\mathbf{b}$, the vector $\mathbf{r}$ from the body to the geocenter, and the barycentric unit vector $\mathbf{s}$ to the quasar Q.

Surprisingly, we found that the first term in Formula (2) is equal to the term including the PPN parameter γ of the geometric delay, but with opposite sign, and consists of distance r. We want to bring your attention to the fact that, although the distances are different, the terms are equal to within about 0.1 ps. Keeping in mind that $(\mathbf{b}\cdot\mathbf{s} = |\mathbf{b}|\cos\varphi)$, the formula for the total group delay recommended by the IAU [3, 6] becomes:

$$\tau_{group} = \frac{\tau_{grav} - \frac{\mathbf{b}\cdot\mathbf{s}}{c}(1-\frac{(\gamma+1)GM}{c^2 r}+\ldots)}{1+\frac{1}{c}(\mathbf{s}\cdot(\ldots))} = \frac{\tau_{\text{GR}}+\ldots}{1+\frac{1}{c}(\mathbf{s}\cdot(\ldots))}, \tag{3}$$

where τ_{GR} is the resultant contribution of General Relativity (GR) effects to the τ_{group}, including two relativistic terms which cancel each other out. Then, τ_{GR} may be written as follows for $\gamma = 1$

$$\tau_{\text{GR}} = \frac{2GM}{c^3}\ln\frac{|\mathbf{r_1}|+\mathbf{s}\cdot\mathbf{r_1}}{|\mathbf{r_2}|+\mathbf{s}\cdot\mathbf{r_2}} + \frac{2GM(\mathbf{b}\cdot\mathbf{s})}{c^3 r_2} \tag{4}$$

or, from Formulas (2) and (4), as

$$\begin{aligned}\tau_{\text{GR}} = &\frac{2GM}{c^3}\frac{b}{r_2}\frac{\sin\varphi\sin\theta\cos A}{1-\cos\theta} - \frac{GM}{c^3}\left(\frac{b\sin\varphi\sin\theta\cos A}{r_2(1-\cos\theta)}\right)^2 +\\ &+\frac{GM}{c^3}\frac{b^2}{r_2^2}\frac{1-\cos^2\varphi\cos^2\theta}{1-\cos\theta}\end{aligned} \tag{5}$$

—the total effect of GR in the group delay.

4.1 *The Approximation of Small Angles*

Given that $\gamma = 1$ in GR and ignoring the minor terms of $O(\frac{b^2}{r^2})$ for the sake of simplicity, then

$$\tau_{\text{GR}} = \frac{2GM}{c^3}\frac{b\sin\varphi\sin\theta\cos A}{r_2(1-\cos\theta)} \underset{\theta\to 0}{=} \frac{4GM}{c^3}\frac{b\sin\varphi\cos A}{R}, \tag{6}$$

where $R = \theta\cdot r_2$ is the linear impact parameter. It is now easy to note that Formula (6) corresponds to the formula of the light deflection developed by Einstein in 1916: $\alpha'' = \frac{4GM}{c^2 R}$.

The light deflection angle α'' and τ_{GR} are linked by

$$\tau_{\text{GR}} = \alpha''\frac{b}{c}\sin\varphi\cos A. \tag{7}$$

For an arbitrary angle θ, α'' is

$$\alpha'' = \frac{2GM}{c^2 r_2} \frac{\sin\theta}{1-\cos\theta} \quad (8)$$

Formula (8) proves that the deflection angle as measured with geodetic VLBI is independent of the baseline length in a first approximation. Figure 2 shows the modeled curve of the light deflection angle variations for the approach of Jupiter to the radio source 1922–224. This arc is common for all four baselines (Hobart26–Parkes, Hobart26–Tsukub32, Tsukub32–Kokee, and Parkes–Tsukub32).

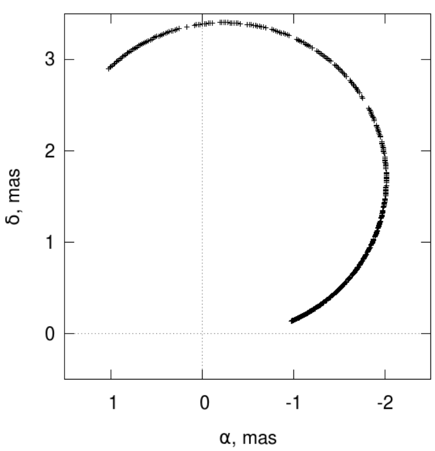

Fig. 2 The light deflection angle α''_A for the baselines Hobart26–Tsukub32, Parkes–Tsukub32, Hobart26–Parkes, and Tsukub32–Kokee for Jupiter to quasar 1922–224.

We can present the resultant contribution of GR in the approximation of small angles to the total group delay in terms of b/R as follows:

$$\tau_{\text{GR}} = \frac{4GM}{c^3}\left[\frac{b}{R}\sin\varphi\cos A - \frac{b^2}{2\,R^2}\sin^2\varphi\cos 2A\right] \quad (9)$$

4.2 Major Term t_1 and Effect of Minor Terms t_2 and t_3

The alternative Formula (5) for an arbitrary angle θ consists of three terms: the term t_1 from Formula (2) gives a major contribution to the GR effects, while the terms t_2 and t_3 from Formula (2) are much smaller than t_1 for the case of the Sun. However, for a very close approach of Jupiter to radio sources (less than 30″) all three terms become comparable.

The four curves of Figure 3 reflect the variations for four VLBI baselines of different length: Kokee–Tsukub32, HartRAO–Wettzell, Onsala60–Wettzell, and HartRAO–Onsala60. Figure 3 shows the dependence of the term t_1 on Universal Time (UT) and the angle θ for the approach of the Sun to the radio source 1243–072. This term varies steadily due to the relatively slow apparent motion of the Sun.

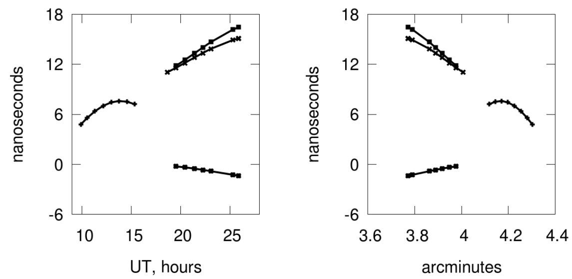

Fig. 3 Major term t_1 vs. UT (left) and vs. angle θ (right) for the case of 1243–072 for the baselines Kokee–Tsukub32, HartRAO–Wettzell, Onsala60–Wettzell, and HartRAO–Onsala60.

The approach of Jupiter to the radio source 1922–224 is more interesting due to the small minimum angular distance. The angle θ increases from 1.4′ to about 5′ rather quickly (in about 12 hours) followed by a fast change in the deflection angle (Figure 2). Figure 4 shows the variations of the term t_1 depending on UT and θ during this event. This term reaches its maximum near 12 UT and becomes negligible over a short period of time.

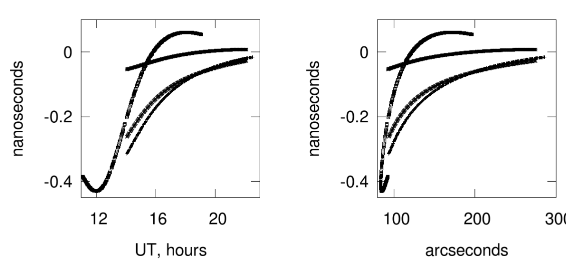

Fig. 4 (left) Major term t_1 from date; (right) t_1 from θ for Jupiter to the radio source 1922–224 for the baselines Hobart26–Tsukub32, Parkes–Tsukub32, Hobart26–Parkes, Tsukub32–Kokee.

Figure 5 shows variations of the sum of the two minor terms (t_2+t_3) depending on UT and θ for the case of 1922–224. Although the angle θ is larger than $1'$, we clearly see wide swings in amplitude in the small terms. In accordance with Formulas (5) and (9), these terms are proportional to $(\frac{b}{R})^2$ (Figure 6); therefore, this sum is becoming very large for longer baselines (Hobart26–Tsukub32) with respect to shorter ones (Hobart26– Parkes) even at the same impact parameter θ. For a very close approach (less than $30''$) the sum (t_2+t_3) for long baselines will be of the same order of magnitude as the major term t_1.

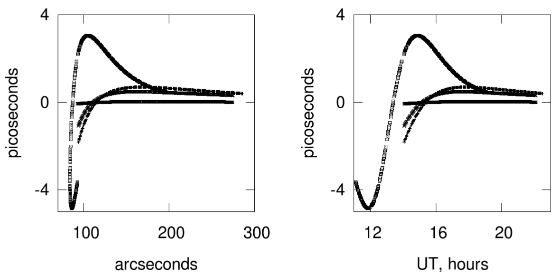

Fig. 5 Variations of the sum (t_2+t_3) for the approach of Jupiter to the radio source 1922–224 vs. θ (left) and vs. UT (right) for the baselines Hobart26–Tsukub32, Parkes–Tsukub32, Hobart26-Parkes, and Tsukub32–Kokee.

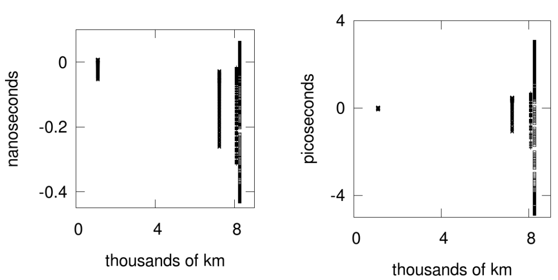

Fig. 6 Variations of the term t_1 (left) and the sum t_2+t_3 (right) for the approach of Jupiter to the radio source 1922–224 vs. baseline length for the baselines Hobart26–Tsukub32, Parkes–Tsukub32, Hobart26–Parkes, and Tsukub32–Kokee.

5 Comparison of the Two Models

The sum of the conventional gravitational delay model (1) and the GR coordinate term $\frac{2GM(\mathbf{b}\cdot\mathbf{s})}{c^3 r_2}$ from the geometric delay can be approximated by Formula (5). Figure 7 shows the variations of the coordinate term and the difference between models (4) and (5) with respect to UT for the approach of the Sun to the radio source 1243–072. This coordinate term does not exceed 1 ns even for a small angle θ.

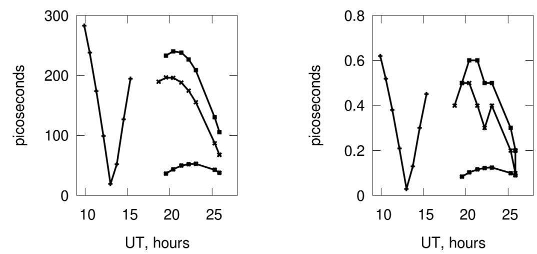

Fig. 7 Variations of the coordinate term (left) and comparison of the two models for the approach of the Sun to the radio source 1243–072 for baselines Kokee–Tsukub32, HartRAO–Wettzell, Onsala60–Wettzell, and HartRAO–Onsala60.

Figure 8 shows the same values as Figure 7 but for the case of 1922–224. The coordinate term here is also negligible. The discrepancies between models (4) and (5) do not exceed 0.1 ps and can be ignored.

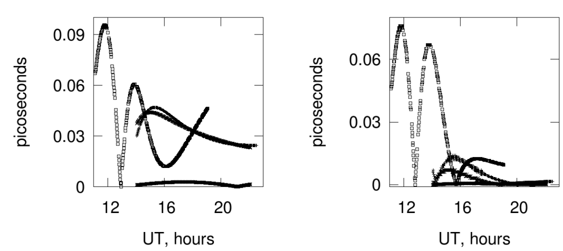

Fig. 8 Variations of the coordinate term (left) and comparison of the two models for the approach of the Jupiter to the radio source 1922–224 for baselines Hobart26–Tsukub32, Parkes–Tsukub32, Hobart26–Parkes, and Tsukub32–Kokee.

6 Conclusion

We proposed an alternative presentation of the effect of GR in the total VLBI group delay model. We showed a new formula, which combines the GR effects from the gravitational and geometric delays with a precision of as much as 1 ps along all ranges of the angular distance between a gravitational body (Sun, Jupiter) and an encountered radio source. In addition, this alternative formula could be easily linked to the light deflection angle at an arbitrary angular distance.

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